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THE KOBAYASHI PSEUDOMETRIC ON ALGEBRAIC MANIFOLDS OF GENERAL TYPE AND IN DEFORMATIONS OF COMPLEX MANIFOLDS

BY

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ABSTRACT. This paper deals with regularity properties of the infinitesimal form of the Kobayashi pseudo-distance. This form is shown to be upper semicontinuous in the parameters of a deformation of a complex manifold. The method of proof involves the use of a parametrized version of the Newlander-Nirenberg Theorem together with a theorem of Royden on extending regular mappings from polydiscs into complex manifolds. Various consequences and improvements of this result are discussed; for example, if the manifold is compact hyperbolic the infinitesimal Kobayashi metric is continuous on the union of the holomorphic tangent bundles of the fibers of the deformation. This result leads to the fact that the coarse moduli space of a compact hyperbolic manifold is Hausdorff. Finally, the infinitesimal form is studied for a class of algebraic manifolds which contains algebraic manifolds of general type. It is shown that the form is continuous on the tangent bundle of a manifold in this class. Many members of this class are not hyperbolic.

Introduction. Let M be a complex manifold, Δ the unit disk in $\mathbf{C}$, and TM the bundle of holomorphic tangent vectors to M. The infinitesimal form F_M on TM of the Kobayashi pseudo-distance on M is defined as: if $\langle x,\xi\rangle \in TM$, then

$$F_M(\langle x, \xi\rangle) = F_M(x, \xi) = \inf R^{-1}$$

where the infimum is taken over all R such that there exists a holomorphic mapping f of the disc of radius R in $\mathbf{C}$ into M with $f_*(\partial/\partial z|_0) = \langle x, \xi\rangle$. This form was first studied by Royden in [13] where it was proved that F_M is upper semicontinuous on T_M.

Using this form, we may define a pseudo-distance on M by

$$d_M(p,q) = \inf_\sigma \int F_M(\sigma(t),\dot\sigma(t))\, dt,$$

where the infimum is taken over all piecewise smooth curves from p to q. The Kobayashi pseudo-distance is defined by

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$$\bar{d}_M(p,q) = \inf \frac{1}{2} \sum_{i=1}^{n} \log \frac{1+|a_i|}{1-|a_i|},$$

where the infimum is taken over all finite sets $\{a_i\}$ in Δ such that there exist n holomorphic mappings f_i from Δ into M with $f_1(0) = p$, $f_i(a_i) = f_{i+1}(0)$ and $f_n(a_n) = q$. Here n is an arbitrary positive integer.

It is well known that $\bar{d}_M$ has the property that it dominates any other pseudo-distance s on M which is distance decreasing with respect to holomorphic maps of Δ into M; that is, $s(f(x),f(y)) \leqslant \bar{d}_\Delta(x,y)$ whenever $f\colon \Delta \to M$ is holomorphic. From this it easily follows that $d_M \leqslant \bar{d}_M$. The opposite inequality is established by Royden [13]. Thus F_M is useful in studying $\bar{d}_M$. It is also easy to see that

$$\bar{d}_M(p,q) = \inf_\sigma L_{\bar{d}}(\sigma),$$

where $L_{\bar{d}}(\sigma)$ is the length of σ with respect to $\bar{d}_M$.

Definition. A complex manifold M is *hyperbolic* if and only if $\bar{d}_M(p,q) \neq 0$ whenever $p \neq q$.

Remarks. (1) It can be shown, see for instance [13], that M is hyperbolic if and only if F_M satisfies the following condition: for every $p \in M$ there exist a coordinate neighborhood U of p and a constant $C = C_p > 0$ such that $F_M(y,\eta) > C\|\eta\|$ for all $\langle y,\eta\rangle \in TM|U$. (Here $\|\eta\|$ can be defined with respect to a hermitian metric or with respect to a norm provided by the coordinate system on U.) If C can be chosen independent of p, we shall say that M is *uniformly hyperbolic*. (2) Using the characterization of hyperbolicity given in (1), one can easily show that M is a *compact* hyperbolic manifold if and only if

$$\sup_{\mathrm{Hol}(\Delta, M)} \|f'(0)\| < \infty$$

where $\|\ \|$ is with respect to any given hermitian metric on M. ($\mathrm{Hol}(X,Y)$ denotes the holomorphic mappings from X to Y.)

An *integrable almost complex structure* on M close to the one provided by the complex structure of M can be thought of as a C^∞ TM-valued (0,1) form φ on M such that $\bar{\partial}\varphi - [\varphi, \varphi]/2 = 0$ and such that all the coefficients of φ are sufficiently small: we shall be using Sobolev norms on M, set up for example as in [9]. By a *deformation of M* we shall mean a set $\mathfrak{F} = \{\varphi(s)|s \in S\}$ of such integrable almost complex structures parametrized by an analytic set S, with $\varphi(s)$ depending smoothly on s and $\varphi(p)$ for some $p \in S$ giving the complex structure of M. The Newlander-Nirenberg Theorem asserts that M can be given a structure as a complex manifold M_s such that $TM_s = \{X - \varphi(s)X | X \in TM\}$. We can obtain a bundle mapping, denoted $\Phi(s)$, from TM

to TM_s (all the TM_s are to be thought of as C^∞ subbundles of $\mathbf{C}T|M|$, the complexification of the tangent bundle of $|M|$, the underlying C^∞ manifold to M): we define $\Phi(s)\xi = \xi - \overline{\varphi(s)}\xi$ for any $\xi \in TM$. Let $F_s = F_{M_s}$.

The basic theorem we prove here is

THEOREM 1. *Let $\langle x, \xi\rangle \in TM$ and $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that if $|s| < \delta$, then $F_s(\langle x,\Phi(s)\xi\rangle) \leqslant F_0(\langle x,\xi\rangle) + \varepsilon$, and in fact, this inequality holds for all $\langle y,\eta\rangle$ in a neighborhood of $\langle x,\Phi(s)\xi\rangle$ in TM_s.*

REMARK. The proof will not depend on the existence of a smooth family and could be phrased to conclude: There is a $\delta > 0$ such that if M_φ is a complex structure on M represented by a C^∞ TM – (0, 1) form φ and $\|\varphi\|$ (Sobolev norm) $< \delta$, then $F_\varphi(\langle x, \Phi(\xi)\rangle) \leqslant F_0(\langle x, \xi\rangle) + \varepsilon$, and in fact, there is a neighborhood of $\langle x, \Phi(\xi)\rangle$ in TM_φ for which the inequality remains true.

The proof of Theorem 1, together with some corollaries concerning the dependence or lack of dependence of δ on ε and $\langle x,\xi\rangle$, occupies the first part of §1. At the end of that section we couple Theorem 1 with some results of R. Brody [1] and show that if M is compact hyperbolic then F_{M_s} is continuous in s (as well as in the fibre direction). This is enough to show that $\bar{d}_{M_s}$ is continuous in s as well. §2 explains how this continuity has implications about the moduli of M. In §3 we discuss the continuity of F_M for M a compact algebraic manifold of general type (actually, M can be assumed to have an *a priori* more general property; see Theorem 8). The last section contains a discussion of the proof of a modified Newlander-Nirenberg Theorem which is needed in §1.

We note that the proof of Theorem 1 can be carried over to establish the upper semicontinuity of intermediate dimensional analogues of F_M, as defined for example in [3].

1. A. **Proof of Theorem 1.** Since the statement of Theorem 1 is local in TM, the proof we shall give will work when M is noncompact; for we may always assume that we are concerned only with a relatively compact part of M, and thus that there are no problems in introducing suitable norms on almost complex structures.

Suppose now that $\langle x,\xi\rangle$ and ε are given. Let $A = F_0(x,\xi)$. Let g be a complex analytic mapping $\Delta_R \to M_0$ such that $g_*(\langle 0,\partial/\partial z\rangle) = \langle x,\xi\rangle$ and $A + \varepsilon > R^{-1} \geqslant A$. What we are going to do is deform g to a mapping g_s of a slightly smaller disc into M_0 which is analytic with respect to the complex structure of M_s.

Define $\mathrm{gr}(g) = \Delta_R \to M_0 \times \Delta_R$ by $\mathrm{gr}(g)(x) = (g(x),x)$. This is an embedding of Δ_R into $M_0 \times \Delta_R$. Set $N_s = M_s \times \Delta_R$. Let $r < R$ be such that we still have $A + \varepsilon > r^{-1} > A$. By Royden's Extension Theorem [14] there exists an equidimensional mapping $G\colon \Delta_r \times \Delta_1^n \to N_0$ such that G is an embedding and such that $G|\Delta_r \times (0, \ldots, 0) = \mathrm{gr}(g)$. Set $D_r = \Delta_r \times \Delta_1^n$. Notice that

(i) $$\mathrm{gr}(g)_*(\langle 0, \partial/\partial z\rangle) = \langle x, \xi\rangle \oplus \langle 0, \partial/\partial z\rangle,$$

and

(ii) $$G_*\big(\langle 0, (\partial/\partial z^1, 0, \ldots, 0)\rangle\big) = \langle x, \xi\rangle \oplus \langle 0, \partial/\partial z\rangle.$$

Let $\{U_j\}$ be a coordinate covering of M_0 with corresponding coordinate z. Then we can express $\phi(s)$ on U_j as

$$\varphi(s) = \sum_{i,k} \varphi^i_{j,k}(s) \frac{\partial}{\partial z^1} d\bar{z}^k.$$

N_0 can be covered by the open sets $V_j = U_j \times \Delta_R$, and these provide coordinates on N_0. Relative to this covering, $\varphi(s)$ defines a family of almost complex structures on N_0 satisfying the integrability conditions there. The biholomorphic mapping G: $D_r \to N_0$ supplies another coordinate system on N_0. Instead of deforming the original mapping g, we are able to deform G; then when we restrict this deformation in the correct way, a deformation of g will be obtained. Now let the standard coordinates on D_r be $(w^1, \ldots, w^{n+1})$. For each s, $\varphi(s)$ is a C^∞ section of the vector bundle $TM_0 \otimes \overline{TM_0^*}$, which naturally determines a section of $TN_0 \otimes \overline{TN_0^*}$. We shall without risk of confusion call this section $\varphi(s)$ also. Under the change of coordinates supplied by G, we have, for instance (suppressing the subscript j referring to the coordinates z_j on V_j),

$$\varphi^i_k(s) \frac{\partial}{\partial z^i} d\bar{z}^k \mapsto \varphi^i_k \sum_{p,q} \frac{\partial w^p}{\partial z^i} \frac{\partial z^k}{\partial w^q} d\bar{w}^q.$$

Here $\partial z^k/\partial w^q$ means $\partial G^k/\partial w^q$ and $\partial w^p/\partial z^i$ means $\partial (G^{-1})^p/\partial z^i$. Thus if we denote by $\{\varphi(s) \circ G\}$ the family of complex structure tensors we obtain on D_r by representing $\{\varphi(s)\}$ in terms of the standard coordinates on D_r, we have

$$\varphi(s) \circ G = \sum_{i,j=1}^{n+1} \theta^i_j(s) \frac{\partial}{\partial w^1} d\bar{w}^j$$

where

(1.1) $$\theta^i_j(s) = \sum_{p,q=1}^{n} \varphi^p_q(s) \frac{\partial w^i}{\partial z^p} \overline{\frac{\partial z^q}{\partial w^j}}.$$

This calculation merely amounts to a verification of the fact that the transition matrices for $TN_0 \otimes \overline{TN_0^*}$ are the Kronecker product of those for the factors TN_0 and $\overline{TN_0^*}$. If M is a transition matrix for the former, then $\overline{{}^tM^{-1}}$ will be the corresponding one for the latter.

We need a lemma the proof of which will be discussed in §4.

LEMMA 1.2. *Let $\eta > 0$ and let $D' \subset\subset D_r$ be any smaller polydisc containing the origin in D_r. Then there is a $\delta' > 0$ such that if $\mu = \sum \mu_j^i (\partial/\partial w^i) d\overline{w}^j$ is an integrable almost complex structure on D_r with $\|\mu\|_k$ or $\|\mu\|_{k+\alpha} < \delta'$ (where k is sufficiently large and depends only on the dimension of D_r), then there is a diffeomorphism $\Psi_\mu: D' \to D_r$ such that*

(i) $\Psi_\mu(0) = 0$;

(ii) Ψ_μ *is holomorphic with respect to the complex structure* μ *on* D_r;

(iii) $\|\Psi_{\mu,*}(x) - I\| < \eta$ *for all* $x \in D'$, *where* $\|\ \|$ *is the maximum of the components and* I *is the identity matrix*;

(iv) $\|\Psi_\mu - \mathrm{id}\|_k \to 0$ *as* $\|\mu\|_k \to 0$. (*The norms* $\|\ \|_k$ *and* $\|\ \|_{k+\alpha}$ *are the Sobolev and Hölder norms, respectively*.)

Here is the situation to which we apply the lemma:

$$D' \subset \underset{\{\varphi(s)\circ G\}}{D_r} \xrightarrow{G} \underset{\{\varphi(s)\}}{N_0} = M_0 \times \Delta_R.$$

Let $D' = \Delta_{r'} \times \Delta_{1/2}^n$ where $A + \varepsilon \geqslant 1/r' > A$ and η is unspecified for the moment. Then we obtain δ' so that if $\|\varphi(s) \circ G\|_k < \delta'$ we have mappings $\Psi_s = \Psi_{\varphi(s)\cdot G}$. Pick δ so that $|s| < \delta$ implies that $\|\varphi(s) \circ G\|_k < \delta'$. Hence for $|s| < \delta$ we obtain

$$D' \xrightarrow{\Psi_s} D_r \xrightarrow{G} N_0$$

such that Ψ_s is holomorphic with respect to the structure $\varphi(s) \circ G$ on D_r. Thus $G_s = G \circ \Psi_s: D' \to N_s = M_s \times \Delta_R$ is holomorphic. Let $\pi_s: M_s \times \Delta_R \to M_s$ be the projection. Denote by τ_1 the vector in $(TD')_0$ with coordinates $(1,0,\ldots,0)$. Since $F_{D'}$ is continuous (see "Calculation" below), there is a neighborhood V of $\langle 0,\tau_1\rangle$ in TD' such that for all $\langle s,\sigma\rangle \in V$,

$$F_{D'}(s,\sigma) \leqslant F_{D'}(0,\tau_1) + \varepsilon = 1/r' + \varepsilon \leqslant A + 2\varepsilon.$$

Put $U_s = (\pi_s \circ G_s)_*(V) \subset TM_s$. This will contain an open subset of TM_s, since all mappings have rank at least n. What we have so far is the following result: If $|s| < \delta$, then for all $\langle y,\eta\rangle \in U_s$, there exists $(s,\sigma) \in V$ such that

$$F_s(y,\eta) = F_s\big((\pi_s \circ G_s)_*(s,\delta)\big) \leqslant F_{D'}(s,\sigma) \leqslant A + 2\varepsilon.$$

To finish the proof we need to show that after possibly making δ smaller we can insure that $\langle x,\Phi(s)\xi\rangle \in U_s$ for $|s| < \delta$. We do this by noting that as soon as s is small enough $v_s = (G_s^{-1})_*(\langle x,0;\Phi(t)\xi,\partial/\partial z\rangle)$ is in V. This is true since

(1) $(G_s^{-1})_* \to (G_0^{-1})_*$ as $|s| \to 0$ by Lemma 1.2;

(2) $\Phi(s)\xi \to \xi$ as $|s| \to 0$;

(3) $(G_0^{-1})_*(\langle x,0;\xi,\partial/\partial z\rangle) = \tau_1 \in V$. Q.E.D.

REMARKS. (1) Although we have not exhibited in the above proof the g_s promised earlier, it is easy to see $g_s = \pi_s \circ G_s$ restricted to the disc in D' going through the origin and having v_s as its tangent direction.

(2) The δ obtained in the theorem depends on $\langle x,\xi\rangle$ via its dependence on G. Before further discussing this dependence and showing how to get rid of it in some cases, we do a calculation related to the last part of the proof of the theorem.

Calculation. Recall that we wished to show that

$$v_s = (G_s^{-1})_*(\langle x, 0; \Phi(s)\xi, \partial/\partial z\rangle) \in V$$

for s sufficiently small. Let us estimate the distance from $\langle 0, \tau_1\rangle$ to v_s. (We shall also write τ_1 for the vector with the same coordinates in $(TD_r)_0$.) Let $v_s = \langle 0,u_s\rangle \in TD'$. Then

$$\begin{aligned}
\|\tau_1 - u_s\| &= \left\|\tau_1 - (G_s^{-1})_{*,(x,0)}(\Phi(s)\xi, \partial/\partial z)\right\| \\
&\leqslant \left\|I(\tau_1) - (\Psi_s^{-1})_{*,(x,0)}(\Phi(s)\xi, \partial/\partial z)\right\| \\
&\leqslant \left\|I(\tau_1) - (\Psi_s^{-1})_{*,0}(\tau_1)\right\| \\
&\quad + \left\|(\Psi_s^{-1})_{*,0}(\tau_1) - (\Psi_s^{-1})_{*,0} \circ (G^{-1})_{*,(x,0)}\left(\Phi(s)\xi, \frac{\partial}{\partial z}\right)\right\| \\
&\leqslant \left\|I - (\Psi_s^{-1})_{*,0}\right\| \\
&\quad + \left\|(\Psi_s^{-1})_{*,0}\right\| \left\|\tau_1 - (G^{-1})_{*,(x,0)}\left(\Phi(s)\xi, \frac{\partial}{\partial z}\right)\right\| \\
&\leqslant \eta + (1+\eta)\left\|(G^{-1})_{*,(x,0)}\left(\xi, \frac{\partial}{\partial z}\right) - (G^{-1})_{*,(x,0)}\left(\Phi(s)\xi, \frac{\partial}{\partial z}\right)\right\|.
\end{aligned}$$

Recall that if $\xi = \Sigma\xi^i\partial/\partial z^i$, then

$$\Phi(s)\xi = \xi - \sum_{j,k} \overline{\varphi_k^j}(s)\xi^k \frac{\partial}{\partial \bar{z}^j}.$$

Hence the above expression is

$$\begin{aligned}
&\leqslant \eta + (1+\eta)\left\|(G^{-1})_{*,(x,0)}\right\| \left\|\sum_{j,k} \overline{\varphi_k^j}(s)\xi^k \frac{\partial}{\partial \bar{z}^j}\right\| \\
&\leqslant \eta + (1+\eta)\left\|(G^{-1})_{*,(x,0)}\right\| \max_{j,k} \|\varphi_k^j(s)\| \; \|\xi\|
\end{aligned}$$

(where the norm on φ is the sup norm)

$$\leqslant \varepsilon/2 + 2\left\|(G^{-1})_{*,(x,0)}\right\| \max_{j,k} \|\varphi_k^j(s)\| \; \|\xi\| = L,$$

where we have chosen $\eta = \varepsilon/2$. Now

$$V \cap (TD')_0 = \left\{ \langle 0,\sigma \rangle : \max_{i=2,n+1} \left(\frac{|\sigma_1|}{r'}, \frac{|\sigma_i|}{1/2} \right) \leq \frac{1}{r'} + \varepsilon \right\}.$$

The above calculation shows that $|1 - u_s^1| \leq L$ and $|u_s^i| < L$ for $i = 2, \ldots, n+1$. Hence $v_s \in V$ if (a) $|u_s^1|/r' \leq 1/r' + \varepsilon$ and (b) $2|u_s^i| \leq 1/r' + \varepsilon$, $i = 2, \ldots, n+1$. This will be true if $L < \varepsilon$, or

$$2\left\|(G^{-1})_{*,(x,0)}\right\| \cdot \max_{j,k} \|\varphi_k^j(s)\| \; \|\xi\| < \frac{\varepsilon}{2}\,. \tag{1.3}$$

Equation (1.3) provides a more explicit end to the proof of Theorem 1′.

We now discuss the dependence of δ on $\langle x,\xi \rangle$.

The fact that F is homogeneous in vectors means that we can immediately obtain the following

COROLLARY 1.4. *Let* $\{\varphi(s)\}$, $\langle x,\xi \rangle$ *and* ε *be given as in Theorem* 1. *Then there exists* $\delta > 0$ *such that for all* s *with* $|s| < \delta$, $F_s(x,\Phi(s)\xi) \leq F_0(x,\xi) + \|\xi\|\varepsilon$, *and the same inequality is true for* $\langle y,\eta \rangle$ *in a neighborhood of* $(x,\Phi(s)\xi)$ *in* $TM_s \cdot (\|\xi\| = \max|\xi^i|)$. *So we have trivially been able to reduce the dependence of* δ *to* x *and the direction of* ξ.

REMARKS. (1) Although F is homogeneous, it does not necessarily behave like a norm on the fibre. There is no way in general to relate $F_M(x,\xi), F_M(x,\eta)$, and $F_M(s,\xi + \eta)$.

(2) Recall that δ was chosen to be small enough so that

(a) $\|\varphi(s) \circ G\|_k < \delta'$ if $|s| < \delta$; δ' was supplied from a lemma on the uniform integration of small complex structures on a polydisc, and

(b) $\|\varphi(s)\|_\infty \leq \varepsilon/4\|(G^{-1})_{*,(x,0)}\|$ if $|s| < \delta$, where $\| \ \|_\infty$ represents the sup norm on $G(D_r) \subset N_0$. Referring to equation (1.1), it is clear that $\|\varphi(s) \circ G\|_k \leq K_G\|\varphi(s)\|_k$ (where $\|\varphi(s)\|_k$ is the k norm on $\pi \circ G(D_r)$). Here K_G depends on a certain number of derivatives of G and G^{-1} with respect to the coordinates on D_r and V^i. Thus we see how δ depends on the derivatives of the mapping G provided by Royden's Extension Theorem [14] and extending the original $g\colon \Delta_R \to M_0$. The better control we can achieve over G, the better the theorem we can prove, in the sense of getting rid of the dependence of δ on the point in the tangent bundle.

(3) Before stating improved versions of Theorem 1, we make the obvious remark that the existence of upper semicontinuous behavior of F_M implies the same for F_s as s varies in a trivial deformation of M. So long as such genuine noncontinuous behavior is not ruled out by added assumptions about M, we should not expect to strengthen Theorem 1 to a continuity statement.

When we know that F_{M_0} is continuous on the tangent bundle, we can easily obtain improvements of Theorem 1.

DEFINITION. A hyperbolic manifold is said to be *complete hyperbolic* if M is complete with respect to d_M, i.e., if every Cauchy sequence with respect to d_M converges.

PROPOSITION 2. *Suppose we are given a family of complex structures* $\{\varphi(s): s \in S\}$ *on a complete hyperbolic manifold* M_0. *Let* $\varepsilon > 0$ *and* $x \in M_0$ *be given. Then there is* $\delta > 0$ *and a neighborhood* U *of* x *in* M_0 *such that* $|s| < \delta$ *implies that for all* $\langle y,\eta\rangle \in TM_0|U$, $F_s(y,\Phi(s)\eta) < F_0(\langle y,\eta\rangle) + \|\eta\|\varepsilon$. *As usual, the inequality remains valid for points in a neighborhood of* $\langle y,\Phi(s)\eta\rangle$ *in* TM_s.

PROOF. A complete hyperbolic manifold is *taut*, where taut means that the family of analytic mappings from the unit disc into M_0 is a normal family. Using this fact it can be shown that F_{M_0} is continuous on T_{M_0}. See [13]. We shall use the continuity of F_{M_0} to show that we only have to apply Royden's Extension Theorem a finite number of times to obtain the desired δ. Let SM_x denote the unit tangent vectors at x. For $\xi \in SM_x$ we obtain

$$G_{x,\xi}: D_{x,\xi} \to N_0 = M_0 \times \Delta_{R_{x,\xi}}$$

such that

$$F_0(x,\xi) + \varepsilon > 1/r_{x,\xi} \geqslant F_0(x,\xi),$$

where $D_{x,\xi} = \Delta_{r_{x,\xi}} \times \Delta_1^n$. Let

$$N_{x,\xi} = \{\langle y,\eta\rangle \in TM_0: |F_0(y,\eta) - F_0(x,\xi)| < \varepsilon/100\}.$$

Since $F_{D_{x,\xi}}$ is continuous, there is a neighborhood P_{0,τ_1} of $\langle 0, \tau_1\rangle = \langle 0,(1, 0, \ldots, 0)\rangle$ in $TD_{x,\xi}$ such that if $\langle s, \sigma\rangle \in P_{0,\tau_1}$ then

$$|F_{D_{x,\xi}}(s,\sigma) - F_{D_{x,\xi}}(0,\tau_1)| < \varepsilon/100.$$

Let $V_{x,\xi} = N_{x,\xi} \cap (G_{x,\xi})_*(P_{0,\tau_1})$. Then

$$SM_x \subseteq \bigcup_{\xi \in SM_x} V_{x,\xi} = \bigcup_{i=1,p} V_{x,\xi_i} = V$$

since SM_x is compact. Let $U = \pi(V)$ where π: $TM \to M$ is the projection. Then an easy calculation shows that $\delta = \min_{i=1,p} \delta_i$ will verify the inequality of the proposition for 2ε. Q.E.D.

COROLLARY 2.1. *Suppose* M *is a compact hyperbolic manifold and* $\{\varphi(s)\}$ *is as above. Then given* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that for all* $\langle x,\xi\rangle \in TM$, $|s| < \delta$ *implies that* $F_s(x,\Phi(s)\xi) \leqslant F_0(x,\xi) + \varepsilon\|\xi\|$; *this inequality is valid for* $\langle y,\eta\rangle$ *in a neighborhood of* $\langle x,\Phi(s)\xi\rangle$ *in* TM_s.

In a similar manner one can prove

PROPOSITION 3. (i) *If* F_{M_0} *is continuous on* TM_0, *then we have the same result as in Proposition* 2, *and as in Corollary* 2.1 *if* M_0 *is assumed compact.*

(ii) *If* $F_{M_0}|TM_{0,x}$ *is continuous for some* $x \in M_0$, *then, given* $\varepsilon > 0$, *there is a*

$\delta > 0$ *such that if* $|s| < \delta$, *then for all* $\xi \in TM_{0,x}$, $F_s(x,\Phi(s)\xi) \leqslant F_0(x,\xi) + \varepsilon\|\xi\|$.

(iii) *If* F_{M_0} *is continuous at* $\langle x,\xi\rangle$, *then for* $\varepsilon > 0$ *there is a* $\delta > 0$ *and a neighborhood* U *of* $\langle x,\xi\rangle$ *in* TM_0 *such that if* $\langle y,\eta\rangle$ *is an element of* $\pi^{-1}(\pi(U))$ *and* $|s| < \delta$, *then* $F_s(y,\Phi(s)\eta) \leqslant F_0(y,\eta) + \varepsilon\|\eta\|$ (π: $TM_0 \to M_0$ *is the projection*).

REMARK. In Proposition 3 all the inequalities hold true for neighborhoods in the various TM's, as in Proposition 2.

We now give some applications of Theorem 1. Recall that a manifold is uniformly hyperbolic if there exists a C such that $F_M(y,\eta) \geqslant C\|\eta\|$.

DEFINITION. If M is uniformly hyperbolic, let C_M be the supremum of all C's satisfying that definition. We call C_M a *constant of hyperbolicity*. Set $C_M = 0$ if M is not uniformly hyperbolic. Notice, of course, that M may be hyperbolic and have $C_M = 0$. In the context of a deformation, set $C_s = C_{M_s}$.

Now let M_0 be a compact complex manifold, not necessarily hyperbolic and $\{\varphi(s)|s \in S\}$ a family of complex structures on M_0. Then as shown by Kuranishi [9], we may suppose that there is a complex space V, a differentiable mapping ω: $V \to S$ of constant rank such that the complex structure on $\omega^{-1}(s)$ is that determined by $\varphi(s)$. Further, we can find a finite covering $\{U_j\}$ of V such that $U_j = D_1 \times S$ (analytically), where D_1 is the unit polydisc in $\mathbf{C}^n$. Using this covering, we can define a norm on tangent vectors to the fibres. Namely, if $\eta \in TM_{s,y}$, then $\|\eta\| = \max_{ij}|\eta_i^j|$ where the η_i^j for $j = 1, \ldots, n$ are the coordinates of the vector η with respect to the coordinate system U_i.

PROPOSITION 4. *If* $\{V,\omega,S\}$ *is such that there exist a sequence* $\{s_i\} \subset S$ *and a positive constant* b *such that* $s_i \to 0$ *and* $C_{s_i} \geqslant b$ *for all* i, *then* M_0 *is hyperbolic (in fact, uniformly hyperbolic).*

PROOF. This follows trivially from the basic upper semicontinuity supplied by Theorem 1. For given $\langle x,\xi\rangle \in TM_0$ with $\|\xi\| = 1$, then as soon as s_i is small enough we have

$$b\|\Phi(s_i)\xi\| \leqslant F_{s_i}(x,\Phi(s_i)\xi) \leqslant F_0(x,\xi) + \varepsilon$$

for any $\varepsilon > 0$. Hence $b\|\xi\| \leqslant F_0(x,\xi)$. Q.E.D.

COROLLARY 4.1. *Suppose* $\{V,\omega,\Delta_r\}$ *is a deformation of a compact manifold* M_0. *If* M_s *is biholomorphic to* M' *for each* s *in a set with limit point zero and* M' *is hyperbolic, then* M_0 *is hyperbolic.*

REMARKS. (1) The reader will have no trouble stating local versions of Proposition 4. One can obtain similar results by making assumptions about the variation of local constants of hyperbolicity.

(2) Recently, in [2], M. Green and R. Brody have given an example of a smooth family of deformations of a nonhyperbolic compact algebraic manifold parametrized by the disc. Corollary 4.1 implies that M_s, $s \neq 0$, in this family cannot be the same complex manifold for s sufficiently small, since they *are* all hyperbolic, as Green and Brody prove.

B. **Continuity of F_{M_s}.** The lower semicontinuity of F_{M_s} in s is still an open question, even when M is compact. An unpublished example of Royden provides a domain D in $\mathbf{C}^2$ such that F_D is discontinuous. See §3.

If M is a compact hyperbolic manifold, then recent results of R. Brody [1] coupled with the above Theorem 1 show that F_{M_s} is continuous in s, for s sufficiently close to a parameter value corresponding to a hyperbolic structure M. Thus a small deformation of a compact hyperbolic manifold is hyperbolic. But Brody's technique does not yield the lower semicontinuity of F_{M_s} for deformations of arbitrary complex manifolds; it does prove that the constant of hyperbolicity, C_{M_s}, is lower semicontinuous in such a situation.

As a consequence of continuity of F_{M_s} in s, we have the following theorem whose proof can be found in the thesis of Brody [1].

THEOREM 5. *Let $\{\varphi(s) | s \in S\}$ represent a family of deformations of a compact complex analytic manifold M. If F_{M_s} is continuous on $\bigcup_{s \in S} TM_s$ then $\bar{d}_{M_s}$ is continuous on $M \times M \times S$.*

The tautness of a compact hyperbolic manifold M implies that F_M is continuous in TM, so we obtain

COROLLARY 5.1. *If $\{\varphi(s) | s \in S\}$ is a deformation of M, and M is compact hyperbolic, then for any sufficiently small neighborhood U of zero in S, $\bar{d}_{M_s}$ is continuous on $M \times M \times U$.*

2. **Moduli for compact hyperbolic manifolds.** If M is compact and hyperbolic, the continuity of $\bar{d}_{M_s}$ can be exploited to give the following:

THEOREM 6. *Let $\{\varphi(s) | s \in S\}$ be a deformation of a compact hyperbolic manifold M. Then if U is any sufficiently small neighborhood of* 0 *in S, the family $A = \bigcup_{s,t \in \bar{U}} \operatorname{Isom}(M_s, M_t)$ is compact. (Here $\operatorname{Isom}(M_s, M_t)$ is the set of all biholomorphisms from M_s to M_t.)*

PROOF. The proof follows closely Narasimhan and Simha, see [10]. Let U be small enough so that M_s for $s \in \bar{U}$ is hyperbolic. Then it is immediate from the continuity of $\bar{d}_s$ on $M \times M \times \bar{U}$ that there exists a constant $B = B_U$ such that

$$B^{-1}\bar{d}_s(x,y) \leq \bar{d}_0(x,y) \leq B\bar{d}_s(x,y) \tag{6.1}$$

for all $s \in \bar{U}$ and $x,y \in M$. Suppose now that f: $M_s \to M_t$ is in A. Then (6.1) implies that

$$\bar{d}_0(f(x), f(y)) \leq B\bar{d}_t(f(x), f(y)) \leq B\bar{d}_s(x, y) \leq B^2\bar{d}_0(x, y).$$

Here the penultimate inequality follows because the Kobayashi distance is distance decreasing with respect to holomorphic mappings between complex manifolds. Thus A is an equicontinuous family, and so by the Arzela-Ascoli Theorem any sequence in A will have a subsequence converging uniformly to a homeomorphism of M. This limit mapping is holomorphic because the total space of the deformation is locally biholomorphically a product and hence Montel's Theorem applies. Q.E.D.

Using precisely the same techniques as in the cited work of Narasimhan and Simha, we may prove the following:

THEOREM 7. *Let M be a compact complex manifold and let $\mathfrak{M}$ be the collection of isomorphism classes of hyperbolic complex structures on M. Then $\mathfrak{M}$ has the structure of a Hausdorff complex space such that if $\{M_s\}_{s\in S}$ is any family of hyperbolic complex structures on M then the map sending s to the isomorphism class of M_s is a morphism from S to $\mathfrak{M}$* [2].

3. **The continuity of F_M.** The form F_M is only upper semicontinuous on TM for unrestricted M. There is an unpublished example due to H. Royden of a domain D in $\mathbf{C}^2$ which is not hyperbolic but such that for each x in D there exists a constant C_x such that $F_D(x,\xi) \geq C_x\|\xi\|$ for all $\langle x,\xi\rangle \in TD_x$.

But there is one class of manifolds for which we can establish the continuity.

DEFINITIONS. If M is a complex analytic manifold and w is a coordinate mapping at p in M let $k_w(p) = \inf\{|\det J_f^w(0)|^{-2}\}$ where the infimum is taken over all holomorphic mappings f from the n-dimensional unit ball B^n into M such that $f(0) = p$. The *hyperbolic volume form* $\eta(p)$ is defined by

$$\eta(p) = n!\,(1/2i)^n k_w(p) dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n.$$

D. Pelles has shown [12] that η is upper semicontinuous and thus defines a measure on M; furthermore he shows that this measure differs from the Kobayashi hyperbolic measure (defined in [5]) by a constant factor depending only on n. A manifold M will be said to be *definite measure hyperbolic* if $\eta(p) \neq 0$ for all p. We shall now prove F_M is continuous for a class of definite measure hyperbolic manifolds, which contains algebraic manifolds of general type (see below for a definition of general type).

THEOREM 8. *Suppose that M is a compact definite measure hyperbolic manifold with canonical bundle K. Assume that for some m there are global*

(2)There is evidence that the same sort of theorem should be true when M is of general type. See [17].

sections $s_1, \ldots, s_k$ of K^m such that $z \mapsto [s_1(z), \ldots, s_k(z)]$ defines a projective embedding of M such that $(s_i\bar{s}_i)^{1/m}/\eta$ is bounded on M for all i. Then F_M is continuous on TM.

PROOF. We only have to show that F_M is lower semicontinuous in the case at hand. Hence let ε be given and suppose $\{f_i: \Delta \to M\}$ is a sequence of holomorphic mappings with $f_{i*}(\gamma_i) = \langle z_i,\xi_i\rangle$, $\langle z_i,\xi_i\rangle$ converging to $\langle z,\xi\rangle$ and $\gamma_i \to \gamma$. We may assume $\gamma \neq 0$. Suppose also that $|\gamma_i| \leqslant F_M(z_i,\xi_i) + \varepsilon$ for all sufficiently large i. For any $r < 1$ we shall find a holomorphic mapping f: $\Delta_{r'} \to M$ with $f_*(\gamma) = \langle z, \xi\rangle$ for $r' < r$ and arbitrarily close to r. This will show that $F_m(z, \xi) \leqslant F(z_i, \xi_i) + \varepsilon$ for all sufficiently large i.

By Royden's Extension Theorem [14], for any $r < 1$ and any i we may find a holomorphic extension F_i of f_i such that F_i maps $B^n(r)$, the ball of radius r in $\mathbf{C}^n$, into M; i.e., $f_i = F_i|B^n(r) \cap \{z|z_2 = \cdots = z_n = 0\}$. Moreover, each F_i is biholomorphic on a neighborhood of $0 \in B^n(r)$. As demonstrated by Yau [17], the hypotheses on M imply that $\{F_i\}$ contains a subsequence which converges to a meromorphic mapping F: $B^n(r) \to M$. Because the F_i's are extensions of the f_i's, and $f_{i*}(\gamma_i) \to \langle z,\xi\rangle$ and $\gamma \neq 0$, $\Delta_r = \{z \in B(r)|z_2 = \cdots = z_n = 0\}$ is not contained in the polar subvariety P_F of F. Hence $\Delta_r \cap P_F = \{a_i\}$, a discrete set of points. By Proposition 2 of [16], $F|\Delta_{r'} \cap B(r^1) - \{a_i\}$ extends to a holomorphic mapping of $\Delta_{r'}$ into M, for $r^1 < r$ and arbitrarily close to r. This is the desired f. Q.E.D.

DEFINITION. If M is an n-dimensional compact algebraic manifold, then M is of *general type* if and only if

$$\limsup_{m \to +\infty} m^{-n} \dim H^0(M, \mathcal{O}(K^m)) > 0.$$

COROLLARY 8.1. *If M is compact algebraic of general type, then F_M is continuous on TM.*

PROOF. Such an M is not necessarily definite measure hyperbolic (see [7]), but the existence of sections of a K^m with the desired properties is established in [6] and this is sufficient to conclude the proof of Theorem 8.

REMARK. We note that because of Corollary 8.1, Proposition 3(i) applies to algebraic general type manifolds.

4. **Complex structures on the polydiscs. Proof of Lemma 1.2.** Here we shall discuss the proof of Lemma 1.2, which was needed in the proof of Theorem 1. For convenience, we restate it here.

LEMMA 1.2. *Suppose D_r is the n-dimensional polydisc of radius r and $D' \subset\subset D_r$ is any smaller polydisc containing $(0, \ldots, 0)$. Let $\eta > 0$. Then there is a $\delta > 0$ such that if μ is an integrable almost complex structure on D_r with $\|\mu\|_k$*

or $\|\mu\|_{k+\alpha} < \delta'$ (*where k is sufficiently large and depends only on n*), *then there exists a diffeomorphism* Ψ_μ: $D' \to D_r$ *such that*

(i) $\Psi_\mu(0) = 0$;

(ii) Ψ_μ *is holomorphic if* D_r *is taken with complex structure* μ *and* D' *is given the standard complex structure*;

(iii) $\|\Psi_{\mu,*}(x) - I\| < \eta$ *for all* $x \in D'$, *where* $\|A\|$ *is the maximum of all components of A and I is the identity matrix*;

(iv) $\|\Psi_\mu - \mathrm{id}\|_k \to 0$ *as* $\|\mu\|_k \to 0$.

This lemma follows immediately from work of Richard Hamilton [4] and J. J. Kohn [8]. However, the best proof can be based on the original proof that almost complex structures satisfying the integrability condition determine complex structures, that is, the proof given by Newlander and Nirenberg in [11]. This method for the proof of the lemma is more natural than those based on the solution of the $\bar{\partial}$-Neumann problem. We mean this in the following sense. Namely, recall that in the proof of Theorem 1, we begin with a holomorphic mapping g: $\Delta_R \to M$ *into* the manifold M. Given $R' < R$ and a family $\{\mu(s)|s \in S\}$ of complex structures on M, we want to associate to g a family of deformations $\{g_s: \Delta_{R'} \to M\}$ such that g_s is holomorphic with respect to structure $\mu(s)$ on M. In the original Newlander-Nirenberg proof of the result that an almost complex structure on the polydisc D_r satisfying the integrability condition determines a complex structure, a similar situation is made the starting point; viz., the identity mapping i: $D_r \to D_r$ is to be deformed to a mapping that is holomorphic with respect to the given almost complex structure on the *image* polydisc. The inverse of this deformation (wherever it is defined) provides the desired coordinates which determine the almost complex structure. So the "maps are going in the same direction" in our application in Theorem 1 and in the original approach to finding coordinate functions for integrable almost complex structures. In the approaches based on the $\bar{\partial}$-Neumann problem, the original coordinates themselves are actually deformed.

The proof of Lemma 1.2 is obtained straightforwardly from the proof of the Newlander-Nirenberg Theorem by noting that all estimates needed in their proof may be guaranteed by restricting $\|\mu\|_k$, the norm of the almost complex structure tensor, instead of restricting the size of the domain polydisc. Details of this modification, as well as discussion of the other approaches to proving Lemma 1.2, can be found in [15].

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